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On Best Approximation in L_p Spaces

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1. INTRODUCTION

Let $L_p = L_p(S, \Sigma, \mu)$, $1 \leqslant p < \infty$, be the Banach space of all μ-measurable extended real valued functions (equivalence classes) y on S such that

$$\|y\| = \|y\|_p = \left[\int_S |y(s)|^p \mu(ds)\right]^{1/p} < \infty,$$

where (S, Σ, μ) is a positive measure space. If X is a convex closed non-empty subset of L_p, then an element z in X is called a best approximation to an element y in L_p if

$$\|y - z\| \leqslant \|y - x\|, \tag{1.1}$$

for all x in X. We have proved in [5, 6] that there exists a positive constant $c_p \leqslant 1$ independent of the element y in L_p, $2 \leqslant p < \infty$, such that the strong unicity inequality

$$\|y - z\|^p \leqslant \|y - x\|^p - c_p \|z - x\|^p \tag{1.2}$$

holds for all x in X. The largest constant in (1.2) is

$$c_p = (1 + t_0^{p-1})(1 + t_0)^{1-p} = (p-1)(1 + t_0)^{2-p}, \tag{1.3}$$

where $t_0 = t_0(p)$ denotes the unique zero of the function

$$g(t) = -t^{p-1} + (p-1)\,t + p - 2 \tag{1.4}$$

in the interval $(1, \infty)$ for $p > 2$, and $t_0(2) = 1$.

In this paper we establish a counterpart of (1.2) for L_p spaces where $1 \leqslant p < 2$.

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2. The Main Results

At first, we prove an auxiliary lemma.

Lemma 2.1. *The inequality*

$$p|u|^{p-2}u(v-u) \geqslant |v|^p - |u|^p - c|v-u|^p; \qquad 1 \leqslant p < 2,\ c > 1, \tag{2.1}$$

holds for all $u, v \in \mathbb{R}$ *such that* $u \neq 0$ *when* $p = 1$ *if and only if* $c \geqslant c_p$ *where* $c_1 = 2$ *and* c_p *is as in* (1.3) *for* $1 < p < 2$.

Proof. At first, we suppose that $p = 1$ and $u \neq 0$. Then the inequality (2.1) can be written in the form

$$|v| - v \operatorname{sgn}(u) \leqslant c|v-u|.$$

If $v = 0$ or $\operatorname{sgn}(u) = \operatorname{sgn}(v)$ then the inequality holds for any $c > 0$. Otherwise, if $\operatorname{sgn}(u) = -\operatorname{sgn}(v) \neq 0$ then we have

$$|v| - v \operatorname{sgn}(u) = 2|v| \leqslant 2|v-u| \leqslant c|v-u|,$$

for any constant $c \geqslant 2$. Since u can be arbitrarily close to zero, it follows that the smallest constant c in the inequality (2.1) is equal to 2.

Now, suppose that $1 < p < 2$. By the definition of t_0 it follows that

$$t_0^{p-1} = (p-1)\, t_0 + p - 2.$$

This implies the second equality in (1.3). By the first formula for c_p, we have $c_p > 1$. If $u = 0$ then the inequality (2.1) is true for all $v \in \mathbb{R}$ and $c \geqslant 1$. In particular, this holds for all $c \geqslant c_p$. Consequently, we can assume that $u \neq 0$ and denote $t = v/u \in \mathbb{R}$. Dividing both sides of the inequality (2.1) by $|u|^p$, we get the equivalent inequality

$$f(t) = f(t, c) := |t|^p - c|t-1|^p - pt + p - 1 \leqslant 0.$$

Since $f''(0) = -f''(1) = +\infty$ and

$$f''(t) = p(p-1)[|t|^{p-2} - c|t-1|^{p-2}]; \qquad t \neq 0, 1,$$

it follows that the function f is strictly convex on the interval

$$I(c) = [k/(k-1), k/(k+1)], \qquad k = c^{1/(p-2)} < 1,$$

and it is strictly concave otherwise. Moreover, we have

$$f(1, c) = f'(1, c) = 0, \qquad f''(1, c) = -\infty$$

and

$$f(-t_0, c_p) = f'(-t_0, c_p)/p = g(t_0) = 0,$$

$$f''(-t_0, c_p) = p(p-1)(t_0^{p-2} - 1)/(1+t_0) < 0.$$

Hence the points $t = -t_0 < k(c_p)/(k(c_p) - 1)$ and $t = 1 > k(c_p)/(k(c_p) + 1)$ in $\mathbb{R} \backslash I(c_p)$ are unique maxima of the function $f(\cdot, c_p)$. This implies that $f(t, c_p) \leqslant 0$ for all real t. Finally, $f(t, c)$ is a decreasing function of variable c for every fixed $t \neq 1$. Thus we have

$$f(-t_0, c) > f(-t_0, c_p) = 0, \tag{2.2}$$

for any $c < c_p$. This completes the proof. ∎

The unique zero $t_0 = t_0(p) \in (1, \infty)$ of the function $g(t)$ defined by (1.4) lies in the interval (t_l, t_u) (cf. Fig. 1). An easy computation gives

$$t_l = (p-1)^{1/(p-2)} \quad \text{and} \quad t_u = t_l + (p-1)^{-1}, \quad 1 < p < 2. \tag{2.3}$$

The function $(1 + t^{p-1})(1+t)^{1-p}$ of variable $t \geqslant 1$ is decreasing for any $p \in (1, 2)$. Hence by (1.3), we have

$$(1 + t_u^{p-1})(1 + t_u)^{1-p} < c_p < (1 + t_l^{p-1})(1 + t_l)^{1-p} < 2^{2-p}, \qquad 1 < p < 2. \tag{2.4}$$

In particular, it follows that

$$1 < c_p < 2, \qquad \text{for every } 1 < p < 2,$$

$$\lim_{p \to 1+} c_p = 2 \qquad \text{and} \qquad \lim_{p \to 2-} c_p = 1.$$

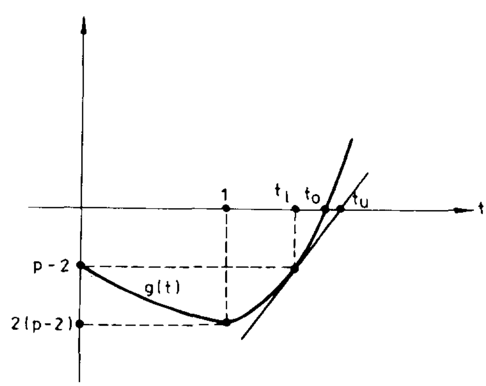

FIG. 1. Lower and upper bounds t_l and t_u for t_0.

The equation $g(t)=0$ can not be solved explicitly for an arbitrary p. However, we easily find that

$$t_0\left(\frac{3}{2}\right)=3+2\sqrt{2}, \qquad c_{3/2}=(1+2^{-1/2})^{1/2}\approx 1.31$$

and

$$t_0\left(\frac{4}{3}\right)=8, \qquad c_{4/3}=3^{1/3}\approx 1.44.$$

Now let $\tau(x, y-x)$ denote the Gateaux derivative of pth power of L_p-norm at the point $x\in L_p$ in the direction $y-x\in L_p$, i.e., let

$$\tau(x, y-x)=\lim_{t\to 0}\frac{\|x+t(y-x)\|^p-\|x\|^p}{t}.$$

It is well known [3, pp. 350–351] that the derivative exists for any x, y in $L_p=L_p(S, \Sigma, \mu)$ $(1<p<\infty)$ and

$$\tau(x, y-x)=p\int_S |x(s)|^{p-2}x(s)[y(s)-x(s)]\,\mu(ds). \tag{2.5}$$

Moreover, if $x\in L_1$ satisfies the condition

$$\mu(N_x)=0;\ N_x:=\{s\in S\colon x(s)=0\},$$

then the Gateaux derivative $\tau(x, y-x)$ exists for any $y\in L_1$ and it is given by the formula (2.5). In the following theorem we establish an inequality for the Gateaux derivative of pth power of L_p-norm $(1\leqslant p<2)$, which seems to be independently interesting. From now on we assume that c_p; $1\leqslant p<2$, is defined as in Lemma 2.1.

THEOREM 2.1. *For any $p\in[1, 2)$, we have*

$$\tau(x, y-x)\geqslant\|y\|^p-\|x\|^p-c_p\|y-x\|^p;\qquad x, y\in L_p, \tag{2.6}$$

where it is additionally assumed that $\mu(N_x)=0$ when $p=1$.

Proof. Apply Lemma 2.1 replacing u by $x(s)$ and v by $y(s)$. Then we obtain

$$p|x(s)|^{p-2}x(s)[y(s)-x(s)]\geqslant|y(s)|^p-|x(s)|^p-c_p|y(s)-x(s)|^p,$$

for every $s\in S\backslash B$, where B is the set of measure zero consisting of all points s in S such that values $x(s)$ or $y(s)$ are not finite. Finally, integrating both sides of this inequality and using (2.5), we get the inequality (2.6). ∎

Remark 2.1. If $p=2$ then in (2.6) we have equality with $c_2=1$. Further, if $p>2$ then in (2.6) the inverse inequality holds [6] with the positive constant $c_p<1$ defined by (1.3).

Remark 2.2. In general, the constant c_p in (2.6) cannot be replaced by a smaller constant. For example, suppose that $1<p<2$ and $\mu(S)<\infty$. If we choose $v=y(s)\equiv -t_0$ and $u=x(s)\equiv 1$, then in view of (2.2) the inequality (2.1) holds only for $c\geqslant c_p$ and it becomes the equality for $c=c_p$. Clearly, this is also true for the inequality (2.6).

Now we present the main result of the paper.

THEOREM 2.2. *Let X be a subspace of $L_p=L_p(S,\Sigma,\mu)$, $1\leqslant p<2$. If $z\in X$ is a best approximation to an element y in L_p, then*

$$\|y-z\|^p\leqslant\|y-x\|^p\leqslant\|y-z\|^p+c_p\|z-x\|^p, \tag{2.7}$$

for all x in X, where it is additionally assumed that $\mu(N_{y-z})=0$ when $p=1$.

Proof. Suppose that $z\in X$ is a best approximation to an element y in L_p and that x is an element in X. Let us replace x and y in the inequality (2.6) with $y-z$ and $y-x$, respectively. Then we get

$$\tau(y-z,z-x)\geqslant\|y-x\|^p-\|y-z\|^p-c_p\|z-x\|^p.$$

By the Kolmogorov criterion [4, p. 90], we have $\tau(y-z,z-x)=0$. This in conjunction with (1.1) completes the proof. ∎

Finally, we present two corollaries which result from Theorem 2.2. For this purpose, we recall [1, p. 222] that the algebraic polynomial $z(t)=t^n-2^{1-n}U_n(t)$ of degree $n-1$, where

$$U_n(t)=\sin(n+1)\,\theta/\sin\theta \qquad (\cos\theta=t,\ -1\leqslant t\leqslant 1)$$

denotes the Chebyshev polynomial of the second kind, is the best approximation to the function $y(t)=t^n$ in the subspace $X=\mathscr{P}_{n-1}$ of all algebraic polynomials of degree less of equal to $n-1$ with respect to the norm of the Lebesgue space $Y=L_1(-1,1)$. Moreover, the error $\|y-z\|$ of this approximation is equal to 2^{1-n}. Hence by Theorem 2.2, we obtain the following corollary.

COROLLARY 2.1. *For every polynomial $w=w(t)$ in $\mathscr{P}_n$ with the coefficient at t^n equal to 1, we have*

$$2^{1-n}\leqslant\|w\|\leqslant 2^{1-n}+2\|w-2^{1-n}U_n\|,$$

where $\|\cdot\|$ is $L_1(-1,1)$-norm and U_n denotes the Chebyshev polynomial of the second kind.

Further, let D_r be the Bernoulli function defined by

$$D_r(t) = \sum_{k=1}^{\infty} k^{-r} \cos(kt + \pi r/2); \qquad 0 \leqslant t \leqslant 2\pi,\ r = 1, 2, \ldots.$$

Then by [2, pp. 61–66] the best $L_1(0, 2)$-approximation z_r to $y = D_r$ in the subspace $X = \mathscr{T}_{n-1} \subset L_1(0, 2\pi)$ of all trigonometric polynomials T of the form

$$T(t) = a_0 + \sum_{k=1}^{n-1} (a_k \cos kt + b_k \sin kt)$$

exists and its error is equal to

$$\lambda_r := \|D_r - z_r\| = 4n^{-r} \sum_{k=0}^{\infty} (-1)^{k(r+1)} (2k+1)^{-r-1}.$$

Hence by Theorem 2.2, we obtain

COROLLARY 2.2. *For every trigonometric polynomial T in* $\mathscr{T}_{n-1}$, *we have*

$$\lambda_r \leqslant \|D_r - T\| \leqslant \lambda_r + 2\|z_r - T\|; \qquad r = 1, 2, \ldots,$$

where $\|\cdot\|$ *denotes the Lebesque* $L_1(0, 2\pi)$*-norm and* z_r *is the best* $L_1(0, 2\pi)$*-approximation to the Bernoulli function* D_r *with the error* λ_r.

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